JOURNAL OF APPROXIMATION THEORY **64**, 214–225 (1991)

Complex Kergin Interpolation

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Communicated by G. Nurnberger

Received November 27, 1989

We show that complex Kergin interpolation may be defind in any domain that is $\mathbb{C}$-convex, whereas the original definition required ordinary, real convexity. We also provide a counterexample which essentially shows that this is the most general definition possible. Finally we give an application concerning approximation of entire functions. © 1991 Academic Press, Inc.

INTRODUCTION

In order to determine a polynomial of degree m in n variables one has to specify its values at $\binom{m+n}{m}$ distinct points. However, if some of the points are allowed to be repeated and if derivatives up to the corresponding orders are also specified at these points, then fewer points, counted with multiplicity, will suffice. The extreme case is of course the Taylor expansion at a point p_0, in which case $m+1$ points (namely p_0 repeated $m+1$ times) are enough to determine the polynomial.

If $n=1$ then the number of points, repeated or not, is always $m+1$, and so there is a unique polynomial of degree m, the Lagrange polynomial, which interpolates the values of a function f at $p_0, p_1, ..., p_m \in \mathbb{R}$.

For arbitrary n, Kergin introduced in [5] a canonical way of choosing a polynomial $\kappa_p f$ of degree m, interpolating a given function f at $p_0, p_1, ..., p_m \in \mathbb{R}^n$. This was done by requiring $\kappa_p f$ also to interpolate derivatives of f of order k somewhere in the convex hull of any $k+1$ of the points. It can be thought of as a non-infinitesimal version of higher-order

0021-9045/91 $3.00

interpolation at repeated points. In one variable this extra interpolation of derivatives is automatic in view of the mean value theorem, and so the Lagrange polynomial is recovered. In fact, $\kappa_p f$ is the unique polynomial of degree m with the property that $f-\kappa_p f$ annihilates the $\binom{m+n}{m}$ linear functionals consisting in integration of derivatives of order k over the convex hull of $p_0, p_1, ..., p_k$, for $k=0, 1, ..., m$. It is also independent of the order of the points.

An explicit formula for $\kappa_p f$ was given in [6]. It is a generalization of the classical one-variable formula

$$\kappa_p f(z)=\sum_{k=0}^{m} [p_0 p_1 \cdots p_k] f\cdot(z-p_0)(z-p_1)\cdots(z-p_{k-1}),$$

with the Newton divided differences $[p_0 p_1 \cdots p_k] f$ replaced by integrals $\int_{[p_0 p_1 \cdots p_k]} D_\alpha f$ of partial derivatives of f over the convex hull of $p_0, p_1, ..., p_k$. In view of the Genocchi formula (cf. [7, p. 16])

$$[p_0 p_1 \cdots p_k] f=\int_{[p_0 p_1 \cdots p_k]} f^{(k)},$$

this does indeed generalize the divided differences.

The Kergin interpolation carries over to the complex case precisely as ordinary Lagrange interpolation. Namely, if f is holomorphic in $\mathbb{C}^n$ and we set $\kappa_p f=\kappa_p(\operatorname{Re} f)+i\kappa_p(\operatorname{Im} f)$, identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, then this complex Kergin polynomial is holomorphic. But whereas ordinary (real) Kergin interpolation is confined to functions defined on convex domains, its complex analogue can be substantially extended and turns out to be closely related to complex convexity.

A domain is said to be $\mathbb{C}$-convex if its intersections with complex lines are all simply connected. In this paper we give a continuous extension of the Kergin interpolation operator to functions holomorphic on an arbitrary $\mathbb{C}$-convex domain. We also provide an example showing that this cannot be done for a general Runge domain.

There is a classical integral formula of Hermite for the remainder $f-\kappa_p f$ which is the key to many interpolation problems in one complex variable. Bloom has obtained similar formulas in balls [2] and polydiscs [3] in $\mathbb{C}^n$. We give here an integral formula in a general $\mathbb{C}$-convex domain with C^2 boundary, which fits nicely into the framework of weighted representation formulas for holomorphic functions.

Using our integral formula we then generalize a one-variable theorem of Gelfond concerning interpolation of entire functions of given type. This has been done by Bloom for the cases where one of the norms $|z|$ or $\max|z_j|$ is used to measure the type of the entire functions. He treats the two cases by quite different methods (cf. the final remark in [3]), whereas with our approach the polydisc case just appears as a limit case.

Complex Convexity

A domain $\Omega \subset \mathbb{R}^n$ is convex if and only if its intersection with an arbitrary real line is a connected interval (or empty); i.e., the intersection is contractible. In the complex case the analoguous concept will involve simple connectedness.

Definition 1. A domain $\Omega \subset \mathbb{C}^n$ is said to be $\mathbb{C}$-convex[1] if its intersection with an arbitrary complex line is contractible (or empty).

Convexity implies $\mathbb{C}$-convexity, because convex sets are contractible. But there are $\mathbb{C}$-convex domains which are not convex. This is obvious for $n = 1$ and below (Example 1) we give examples of bounded $\mathbb{C}$-convex domains in any dimension, even with smooth boundaries, which are not convex.

It is natural also to introduce another, slightly more general, notion of convexity called linear convexity.

Definition 2. A domain $\Omega \subset \mathbb{C}^n$ is said to be linearly convex if its complement is a union of complex hyperplanes.

One can prove (cf. [9, Proposition 1]) that $\mathbb{C}$-convexity implies linear convexity, but linearly convex domains need not be $\mathbb{C}$-convex. Product domains provide counterexamples. However, under mild regularity assumptions the two concepts become equivalent.

Theorem 1. *Let $\Omega \subset \mathbb{C}^n$, $n > 1$, be a bounded domain with C^2 boundary. Then Ω is $\mathbb{C}$-convex if and only if it is linearly convex.*

Proof. See Proposition in [8]. ∎

Example 1. Let Ω be any bounded convex domain with smooth boundary, satisfying $\Omega \cap \{z_1 = 0\} = \varnothing$, and such that $\Omega \cap \{z_2 = \cdots = z_n = 0\}$ contains a segment of the unit circle in its boundary. Denote by $\tilde{\Omega}$ the image of Ω under the mapping

$$(z_1, z_2, ..., z_n) \mapsto \left(\frac{1}{z_1}, \frac{z_2}{z_1}, ..., \frac{z_n}{z_1}\right).$$

[1] The term *strongly linearly convex* is synonymous.

(This mapping just amounts to choosing a new hyperplane at infinity in the complex projective space $\mathbb{P}^n$.) Since hyperplanes get mapped to hyperplanes, linear convexity is preserved. The domain $\tilde{\Omega}$ is thus a bounded $\mathbb{C}$-convex domain with smooth boundary. But the inversion $z_1 \mapsto 1/z_1$ interchanges the interior and exterior of the unit circle, and so $\tilde{\Omega}$ fails to be convex.

We shall make use of the following two properties of $\mathbb{C}$-convex domains.

THEOREM 2. *Let $\Omega \subset \mathbb{C}^n$ be a $\mathbb{C}$-convex domain. Then Ω is a Runge domain.*

Proof. See Proposition 1 in [9]. ∎

THE COMPLEX KERGIN OPERATOR

Let $\Omega \subset \mathbb{C}^n$ be a $\mathbb{C}$-convex domain and $p = (p_0, p_1, ..., p_m)$ a sequence of points in Ω. We denote the standard k-simplex in $\mathbb{R}^k$ by Δ^k, its vertices by $v_0, v_1, ..., v_k$, and for each $k \leqslant m$ we write Ω^k for the intersection of Ω with the complex affine space spanned by $p_0, p_1, ..., p_k$. Also we let $\omega^k \subset \mathbb{C}^k$ be the preimage of Ω^k under the complex affine mapping $\mathbb{C}^k \to \mathbb{C}^n$ taking each v_j to p_j. (We use the canonical inclusion $\mathbb{R}^k \subset \mathbb{C}^k$.) Then ω^k is again $\mathbb{C}$-convex. Finally, we introduce singular chains $\gamma^k\colon \Delta^k \to \omega^k$ mapping every face of Δ^k into the complex $(k-1)$-plane which it spans (this is possible by $\mathbb{C}$-convexity, and it follows that each v_j is fixed), and consider the linear functionals

$$f \mapsto \int_{[p]^k} f \underset{\text{def}}{=} \int_{\gamma^k} f(p_0 + \lambda_1(p_1 - p_0) + \cdots + \lambda_k(p_k - p_0))\, d\lambda_1 \wedge \cdots \wedge d\lambda_k.$$

PROPOSITION 1. *The linear functional*

$$\int_{[p]^k} : \mathcal{O}(\Omega) \to \mathbb{C}$$

defined above is independent of the particular choice of chain γ^k in ω^k.

Proof. Choose a new chain $\tilde{\gamma}^k$ and consider the expression

$$\int_{[p]^k} f - \int_{[\tilde{p}]^k} f = \int_{\gamma^k - \tilde{\gamma}^k} f(\lambda)\, d\lambda.$$

Using the requirement that corresponding faces of γ^k and $\tilde{\gamma}^k$ be mapped into the same complex $(k-1)$-plane, we can find a finite number of new chains δ_j, contained in those planes, so that $\gamma^k - \tilde{\gamma}^k - \sum_j \delta_j$ is a cycle. To be explicit, we take

$$\delta_j = (-1)^j \, \delta_{[0 \cdots \hat{j} \cdots k]},$$

where $\delta_{[j_0 j_1 \cdots j_l]}$ is defined inductively by

$$\begin{cases} \delta_{[j_0]} = 0; \\ \partial \delta_{[j_0 j_1 \cdots j_l]} = \gamma_{[j_0 j_1 \cdots j_l]} - \tilde{\gamma}_{[j_0 j_1 \cdots j_l]} - \sum_\nu (-1)^\nu \, \delta_{[j_0 \cdots \hat{j}_\nu \cdots j_l]}, \end{cases}$$

with $\gamma_{[j_0 j_1 \cdots j_l]}$ being the restriction of γ^k to the complex l-plane spanned by $v_{j_0}, v_{j_1}, ..., v_{j_l}$. For entire f it follows that the integral over $\gamma^k - \tilde{\gamma}^k$ equals the integral over $\Sigma_j \delta_j$, which vanishes because $f(\lambda)\, d\lambda$ is a $(k, 0)$-form. Now, since Ω is a Runge domain the entire functions are dense in $\mathcal{O}(\Omega)$, and the proposition follows. ▌

DEFINITION 3. Let $f\colon \Omega \to \mathbb{C}$ be a holomorphic function. The Kergin polynomial $\kappa_p f$, or simply κf, of f with respect to p is

$$\kappa f(z) = \sum_{k=0}^{m} \sum_{\alpha \in \{1, ..., n\}^k} [p]^k_\alpha f \cdot (z-p)^k_\alpha,$$

where

$$[p]^k_\alpha f = \int_{[p]^k} D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_k} f, \qquad D_j = \partial/\partial z_j,$$

and

$$(z-p)^k_\alpha = (z-p_0)_{\alpha_1} (z-p_1)_{\alpha_2} \cdots (z-p_{k-1})_{\alpha_k}.$$

Remark 1. For $n=1$ we have

$$[p]^k_{1,...,1} f = \int_{[p]^k} f^{(k)} = [p_0 p_1 \cdots p_k] \, f,$$

the classical divided difference of Newton.

Remark 2. If it so happens that f can be continued to a function holomorphic on the convex hull of p, then we can allow the γ^k to be the identity mappings and our formula will coincide with the one given in [6]. This occurs in particular if Ω is itself convex, in which case we thus recover the original Kergin polynomials.

As in [5] we let $P^k(\mathbb{C}^n)$ denote the vector space of complex polynomials of degree $\leqslant k$, and $Q^k(\mathbb{C}^n)$ the complex vector space of constant coefficient differential operators homogeneous of order k.

THEOREM 3. *Let $\Omega \subset \mathbb{C}^n$ be a $\mathbb{C}$-convex domain and $p = (p_0, p_1, ..., p_m)$ a sequence of points in Ω. The Kergin operator $f \mapsto \kappa f$ is the unique linear operator $\mathcal{O}(\Omega) \to P^m(\mathbb{C}^n)$ satisfying*

$$\int_{[p]^k} q(D)(f - \kappa f) = 0, \tag{$*$}$$

for $f \in \mathcal{O}(\Omega)$, $0 \leqslant k \leqslant m$, and $q \in Q^k(\mathbb{C}^n)$.

Moreover,

(i) *it is independent of the order of the points p_j,*

(ii) *it interpolates f at p (including derivatives in the case of multiple points),*

(iii) *it is continuous (in the usual topologies on $\mathcal{O}(\Omega)$ and $P^m(\mathbb{C}^n)$),*

(iv) *it is holomorphic as a function of p,*

(v) *it is invariant under complex affine mappings (i.e., $\kappa_p \Psi^* f = \Psi^* \kappa_{\Psi(p)} f$ for any such mapping $\Psi: \mathbb{C}^n \to \mathbb{C}^l$),*

(vi) *it is a projection onto $P^m(\mathbb{C}^n)$.*

Proof. Properties (iii) and (iv) are quite immediate consequences of the definition.

As for the property (vi), we note that by Remark 2 the restriction of κ to $\mathcal{O}(\mathbb{C}^n)$ is precisely the original Kergin operator, which is known (cf. [6]) to be a projection onto $P^m(\mathbb{C}^n)$.

Since Ω is a Runge domain, the subspace $\mathcal{O}(\mathbb{C}^n)$ is dense in $\mathcal{O}(\Omega)$, and so properties (i), (ii), and (v) follow by continuity from the corresponding ones for Kergin interpolation of entire functions. (Actually only mappings $\Psi: \mathbb{C}^N \to \mathbb{C}$ were considered by Kergin, but the more general invariance is obtained almost as easily.)

Also property ($*$) is a consequence of the continuity of κ and a similar result (cf. [6, proof of Theorem 2]) in the classical case.

It remains to establish the uniqueness of κ. Again, for $\Omega = \mathbb{C}^n$, this was proved by Kergin in [5], and hence we are done if we prove that any linear operator satisfying ($*$) must in fact be continuous. To do so, we write

$$\kappa f(z) = \sum_{|\alpha| \leqslant m} c_\alpha z^\alpha,$$

and

$$\max_{0 \leqslant k \leqslant m} \max_{|\alpha| = k} \max_{\lambda \in \gamma^k} |p_0 + \lambda_1(p_1 - p_0) + \cdots + \lambda_k(p_k - p_0)|^\alpha = M.$$

Then we assume that

$$\max_{0 \leqslant k \leqslant m} \max_{|\alpha| = k} \max_{\lambda \in \gamma^k} \frac{1}{\alpha!} |D^\alpha f(\lambda)| < \delta,$$

which amounts to having f in a typical neighborhood of the origin in $\mathcal{O}(\Omega)$. Now we use the formula

$$\frac{1}{\beta!} D^\beta \kappa f(z) = c_\beta + \sum_{\substack{|\alpha| \leqslant m \\ \alpha > \beta}} \binom{\alpha}{\beta} c_\alpha z^{\alpha - \beta}$$

together with (*) to deduce the estimates

$$|c_\beta| < \delta + \sum_{\substack{|\alpha| \leqslant m \\ \alpha > \beta}} \binom{\alpha}{\beta} |c_\alpha| \cdot M.$$

An induction on $|\beta|$ (from m downwards) yields

$$\max |c_\beta| < \varepsilon$$

provided that $\delta > 0$ is chosen small enough. The continuity is thereby verified. ∎

A natural question that arises in this context is whether or not κ can be extended to a continuous operator $\mathcal{O}(\Omega) \to P^m(\mathbb{C}^n)$ for any Runge domain Ω. The answer is negative, as shown by the following example.

EXAMPLE 2. Take $\varepsilon > 0$ and consider the domains

$$\Omega = \{z \in \mathbb{C}^2; |1 - z_1 z_2| < 1/2, |(z_1 - z_2)(1 - z_1 z_2)| < \varepsilon^2, |z_1| < 2, |z_2| < 2\},$$

and

$$\Omega^\pm = \{|1 - (z_1 \mp i\varepsilon)(z_2 \mp i\varepsilon)| < 1 + \varepsilon^2/2, |(z_1 - z_2) \times (1 - z_1 z_2)| < \varepsilon^2, |z_1| < 2, |z_2| < 2\}.$$

We shall show that there is no continuous extension of the Kergin operator to $\mathcal{O}(\Omega)$, even though Ω (and $\Omega^\pm$ as well) is a Runge domain (cf. [1, Corollary 24.10]).

For ε small enough Ω is contained in both Ω^+ and Ω^-. The intersection of Ω with the complex line $L = \{z_1 = z_2\}$ consists of two components containing the points $p_0 = (-1, -1)$ and $p_1 = (1, 1)$, respectively. These two points are connected in Ω by the curve $[0, \pi] \ni \theta \mapsto -(e^{i\theta}, e^{-i\theta})$. The intersections $\Omega^\pm \cap L$ are connected and we can find curves $\gamma^\pm$ joining p_0 to p_1

in $\Omega^{\pm} \cap L^{\pm}$, where $L^{+}(L^{-})$ denotes the intersection of L with the upper (lower) halfspace $\{\operatorname{Im} z_1 \geqslant 0\}$ ($\{\operatorname{Im} z_1 \leqslant 0\}$).

Now, the function $f(z) = z_1/z_2$ is holomorphic on both Ω^{+} and Ω^{-}. We may thus pick sequences $\{f_\nu^{+}\}$ and $\{f_\nu^{-}\}$ of entire functions converging to f in $\mathcal{O}(\Omega^{+})$ and $\mathcal{O}(\Omega^{-})$, respectively, and hence also in $\mathcal{O}(\Omega)$. Recalling our definition of Kergin interpolation we find that the z_j-coefficient of $\kappa_p f_\nu^{\pm}(z)$ is equal to

$$[p]_j^1 f_\nu^{\pm} = \int_{\gamma^1} D_j f_\nu^{\pm}(-1+2\lambda, -1+2\lambda)\, d\lambda$$

$$= \frac{1}{2} \int_{\gamma^{\pm}} \frac{\partial f_\nu^{\pm}}{\partial z_j}(\tau, \tau)\, d\tau \rightarrow \begin{cases} \mp i\pi/2, & \text{if } j=1; \\ \pm i\pi/2, & \text{if } j=2, \end{cases}$$

as $\nu \rightarrow \infty$.

Summarizing then, we have found sequences $\{f_\nu^{+}\}$ and $\{f_\nu^{-}\}$ both converging to f in $\mathcal{O}(\Omega)$, but such that $\lim_\nu \kappa f_\nu^{+} \neq \lim_\nu \kappa f_\nu^{-}$.

This example can easily be modified so as to have Ω with smooth boundary and, e.g., strictly pseudoconvex.

We end this section by giving an integral formula for the remainder after Kergin interpolation, generalizing the classical formula

$$(f - \kappa f)(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \left(\prod_{j=0}^{m} \frac{z - p_j}{\zeta - p_j} \right) \frac{f(\zeta)\, d\zeta}{\zeta - z}$$

of Hermite.

THEOREM 4. *Let $\Omega \subset \mathbb{C}^n$ be a bounded $\mathbb{C}$-convex domain with C^2 boundary and with a defining function ρ, i.e., $\Omega = \{\rho(z) < 0\}$. Let f be a holomorphic function in Ω continuous up to the boundary, and $p = (p_0, p_1, ..., p_m)$ a sequence of points from Ω. Then the following formula holds*:

$$(f - \kappa f)(z) = \frac{1}{(2\pi i)^n} \int_{\partial\Omega} \sum_{|\alpha| + \beta = n-1} \left(\prod_{j=0}^{m} \frac{\langle \rho'(\zeta), z - p_j \rangle}{\langle \rho'(\zeta), \zeta - p_j \rangle} \right)$$

$$\times \frac{f(\zeta)\, \partial\rho(\zeta) \wedge (\bar{\partial}\partial\rho(\zeta))^{n-1}}{\langle \rho'(\zeta), \zeta - p \rangle^{\alpha} \langle \rho'(\zeta), \zeta - z \rangle^{\beta+1}},$$

where $\alpha = (\alpha_0, \alpha_1, ..., \alpha_m) \in \mathbb{N}^{m+1}$ is a multi-order, $\beta \in \mathbb{N}$, and

$$\rho'(\zeta) = (\partial\rho(\zeta)/\partial\zeta_1, \partial\rho(\zeta)/\partial\zeta_2, ..., \partial\rho(\zeta)/\partial\zeta_n).$$

Proof. Since Ω is linearly convex it follows that every complex tangent plane $T_\zeta = \{z; \langle \rho'(\zeta), \zeta - z\rangle = 0\}$ lies entirely outside Ω. In other words, the mapping $\partial\Omega \times \Omega \ni (\zeta, z) \mapsto \langle \rho'(\zeta), \zeta - z\rangle$ is non-vanishing, and hence (cf. [1, Corollary 3.6]) it gives rise to a Fantappiè formula

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial\Omega} \frac{f(\zeta)\, \partial\rho \wedge (\bar{\partial}\partial\rho)^{n-1}}{\langle \rho', \zeta - z\rangle^n}.$$

The desired formula for $f - \kappa f$ now follows as in [2] by interpolating the kernel function $z \mapsto \langle \rho', \zeta - z\rangle^{-n}$. ∎

This integral formula is in fact a special case of a general representation formula for holomorphic functions (see our forthcoming paper in *Math. Z.*).

Interpolation of Entire Functions

As an application of our integral formula from Theorem 4 we shall generalize results of Gelfond [4, Theorem 2.3.7] and Bloom [2, Theorem 2.3; 3, Theorem 2.3] on interpolation of entire functions.

Definition 4. Let v be a norm on $\mathbb{C}^n$ and λ a positive real number.

(i) Let f be an entire function on $\mathbb{C}^n$. The λ-type of f with respect to v is given by $\tau_v(\lambda) =_{\text{def}} \limsup_{r\to\infty} \log M_v(r)/r^\lambda$, where $M_v(r)$ is the maximum of $|f|$ in the closed ball $v(z) \leqslant r$.

(ii) Let $p = (p_0, p_1, ...)$ be a sequence of points from $\mathbb{C}^n$. The λ-density of p with respect to v is given by $\delta_v(\lambda) =_{\text{def}} \liminf_{r\to\infty} N_v(r)/r^\lambda$, where $N_v(r)$ is the number of points from p in the closed ball $v(z) \leqslant r$.

Theorem 5. *Let f be an entire function on $\mathbb{C}^n$ and $p = (p_0, p_1, ...)$ a discrete sequence of points from $\mathbb{C}^n$, with $v(p_j) \leqslant v(p_{j+1})$, for all j.*

For any complex-homogeneous norm v on $\mathbb{C}^n$ and any positive real number λ, denote by $\tau_v(\lambda)$ the λ-type of f and by $\delta_v(\lambda)$ the λ-density of p, both measured by means of v. Then the inequality

$$\tau_v(\lambda)/\delta_v(\lambda) < c(\lambda) \underset{\text{def}}{=} \int_0^{1/2} \frac{t^{\lambda-1}\, dt}{1-t},$$

ensures that the Kergin polynomials $\kappa_p^m f$ of f with respect to $p^m = (p_0, p_1, ..., p_m)$ converge to f uniformly on compact sets.

Moreover, the constant $c(\lambda)$ is the largest one with this property.

Proof. Let us first assume v to be smooth away from the origin. By Theorem 4 we have the following formula for the remainder after Kergin interpolation at $p^m = (p_0, p_1, ..., p_m)$ with $v(p_m) < R$:

$$(f - \kappa_p^m f)(z) = \frac{1}{(2\pi i)^n} \int_{v(\zeta) = R} f(\zeta)\, T_p^m(\zeta, z),$$

where $v(z) < R$ and

$$T_p^m(\zeta, z) = \sum_{|\alpha| + \beta = n-1} \frac{\langle v'(\zeta), z - p\rangle}{\langle v'(\zeta), \zeta - p\rangle^{\alpha+1}} \frac{\partial v(\zeta) \wedge (\bar{\partial}\partial v(\zeta))^{n-1}}{\langle v'(\zeta), \zeta - z\rangle^{\beta+1}}.$$

We now put $v(p_j) = \rho_j$ and assume $v(z) \leqslant r < \rho_m$. The homogeneity of v implies the Euler formula

$$\mathrm{Re}\langle v'(\zeta), \zeta\rangle = \tfrac{1}{2} v(\zeta),$$

and the convexity translates into

$$\mathrm{Re}\langle v'(\zeta), \zeta - \omega\rangle \geqslant \tfrac{1}{2}(v(\zeta) - v(w)).$$

Writing $\alpha = \overline{\langle v'(\zeta), z - p_j\rangle} / |\langle v'(\zeta), z - p_j\rangle|$ we therefore have the estimates

$$\begin{aligned} |\langle v'(\zeta), z - p_j\rangle| &= \mathrm{Re}\langle v'(\zeta), \alpha(z - p_j)\rangle \\ &= \mathrm{Re}\langle v'(\zeta), \zeta\rangle + \mathrm{Re}\langle v'(\zeta), \alpha(z - p_j) - \zeta\rangle \\ &\leqslant \tfrac{1}{2}(\alpha(z - p_j)) \leqslant \tfrac{1}{2}(r + \rho_j), \end{aligned}$$

and

$$|\langle v'(\zeta), \zeta - p_j\rangle| \geqslant \mathrm{Re}\langle v'(\zeta), \zeta - p_j\rangle \geqslant \tfrac{1}{2}(r - \rho_j) \geqslant \tfrac{1}{2}(r - \rho_m).$$

Now $i^{-n}\, \partial v \wedge (\bar{\partial}\partial v)^{n-1}$ is a positive measure on $\{v = R\}$ with mass $(\pi R)^n$, and it follows that

$$\begin{aligned} &|(f - \kappa_p^m f)(z)| \\ &\quad \leqslant (2\pi)^{-n} M_v(R) \sum_{|\alpha| + \beta = n-1} \frac{(1/2)(r + \rho_0)(1/2)(r + \rho_1) \cdots (1/2)(r + \rho_m)}{(1/2)(R - \rho_0) \cdots (1/2)(R - \rho_m)(1/2)(R - r)} \\ &\quad \times \left(\frac{1}{(1/2)(R - \rho_m)}\right)^{|\alpha| + \beta} (\pi R)^n \\ &\quad = M_v(R) \binom{m + n + 1}{n - 1} \left(\frac{r + \rho}{R - \rho}\right) \left(1 - \frac{\rho_m}{R}\right)^{-n}. \end{aligned}$$

This is the same estimate as the one in [2], and so the argument given there proves the theorem in this case. Let us just briefly indicate where the type and density come into play and how the particular constant turns up. The crucial factors in our estimate are $M_\nu(R)$ and $(r+\rho)/(R-\rho)$. In terms of the counting function N_ν we may easily compute

$$\begin{aligned}\sum_{j=0}^{m} \log\left(\frac{r+\rho_j}{R-\rho_j}\right) &- (m+1)\log\left(\frac{r+\rho_m}{R-\rho_m}\right)\\ &= -\int_0^{\rho_m}\left(\frac{1}{r+t}+\frac{1}{R-t}\right) N_\nu(t)\,dt\\ &= -(r+R)\int_0^{\rho_m}\frac{N_\nu(t)\,dt}{(r+t)(R-t)},\end{aligned}$$

and choosing $R=R(m)=2r+2\rho_m+\varepsilon$ we essentially eliminate the last term in the first expression (the ε is added to compensate the growth of the binomial coefficient $\binom{m+n+1}{n-1}$). If we then were to simply replace $\log M_\nu(R)$ by $\tau_\nu(\lambda)\,R^\lambda$ and $N_\nu(t)$ by $\delta_\nu(\lambda)\,R^\lambda s^\lambda$, where $s=t/R$, we would thus get

$$\log\left[M_\nu(R)\left(\frac{r+\rho}{R-\rho}\right)\right]=\left(\tau_\nu(\lambda)-\delta_\nu(\lambda)\int_0^{\rho_m/R}\frac{(r+R)\,s^\lambda\,ds}{(r+Rs)(1-s)}\right)R^\lambda.$$

And this last integral tends to $c(\lambda)$ as m (and hence R) approaches infinity.

To a general, not necessarily smooth, norm ν (for instance the max-norm corresponding to polydiscs) we can for a given $\varepsilon>0$ find a smooth norm $\tilde\nu$ such that

$$(1-\varepsilon)\nu<\tilde\nu<\nu.$$

From the definitions of type and density it follows that

$$\frac{\tau_\nu(\lambda)}{(1-\varepsilon)^\lambda}\geqslant\tau_{\tilde\nu}(\lambda),$$

$$\delta_{\tilde\nu}(\lambda)\geqslant\delta_\nu(\lambda),$$

and so if ε is taken small enough we will still have

$$\tau_{\tilde\nu}(\lambda)/\delta_{\tilde\nu}(\lambda)<c(\lambda),$$

and we are back to the smooth case.

To see that $c(\lambda)$ is the best possible constant we can simply take the one-variable counterexample of Gelfond [4, Theorem 2.3.7] and consider it as an entire function in $\mathbb{C}^n$, constant in all but one variable, together with a sequence of points from $\mathbb{C} \subset \mathbb{C}^n$. Since the restriction of v to $\mathbb{C}$ coincides up to a constant factor with the ordinary norm, the quotient of type and density remains the same. And since furthermore Kergin interpolation is compatible with restrictions to subspaces it follows that this provides a counterexample in general. ∎

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